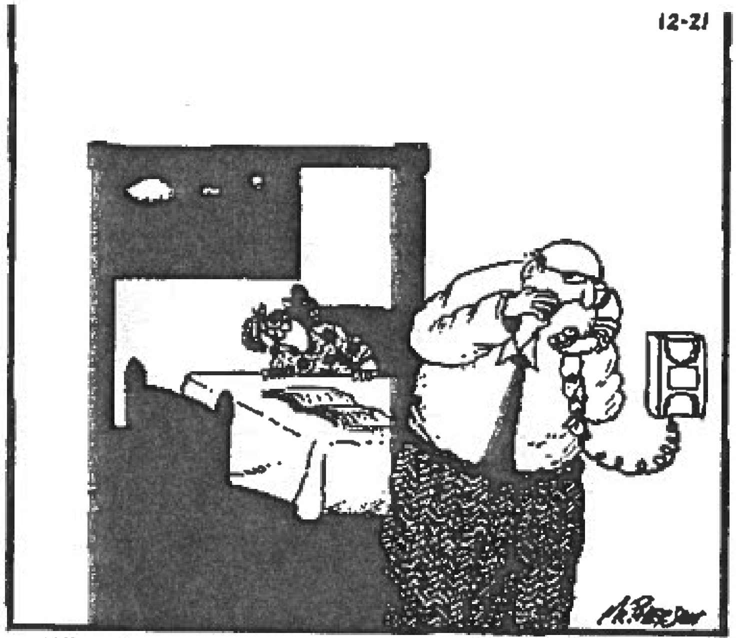
Unit 4: Solving Quadratics



"Uh, yeah, Homework Help Line? I need to have you explain the quadratic equation in roughly the amount of time it takes to get a cup of coffee."

Name: ___________________________ Date: ______________

INTRODUCTION TO POLYNOMIALS

The way we write numbers in our systems is interesting because with only 10 digits, i.e. 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, we are able to write whole numbers as large as we would like. This is because what we really are doing is **counting** how many **powers of 10** that we have.

***Exercise* #1:** Write each of the following numbers as a sum of multiples of powers of 10. The first is done as an example.

(a) $563 = 500 + 60 + 3$
$= 5 \cdot 100 + 6 \cdot 10 + 3$
$= 5 \cdot 10^2 + 6 \cdot 10 + 3$

(b) 274

(c) 3,842

(d) 5,081

(e) 21,478

We can now use algebra to replace the base of 10 with a generic base of x (or whatever variable you like).

***Exercise* #2:** Consider the number 63,735.

(a) As in #1, write this number as the sum of multiples of powers of 10.

(b) If $x = 10$, write this number in terms of an equivalent expression involving x.

The base of a polynomial certainly doesn't have to be 10. But, all polynomials have a form similar to your answer in letter (b). Let's define them a little more definitively.

POLYNOMIAL EXPRESSIONS

Any expression of the form: $ax^n + bx^{n-1} + cx^{n-2} + \cdots + \text{constant}$, where the exponents, $n, n-1$, $n-2$, etcetera are all positive integers. Note that **not all powers** need to be presents because the **coefficients**, i.e. a, b, c, etcetera can be zero.

***Exercise* #3:** Of the expressions shown below, circle all of them that represent polynomials. Discuss why the ones that aren't polynomials fail the definition above.

$4x^2 + 8x + 1$ $\qquad$ $9x^2 + 2x + \frac{1}{x}$ $\qquad$ $2^x + 3^x + 4^x$ $\qquad$ $2x^2 + 5x^3 - x + 8$

2

It is often important to place polynomials in their **standard form**. The standard form of a polynomial is simply achieved by writing it as an **equivalent expression** where the powers on the variables **always descend**.

***Exercise* #4:** Write each of the following polynomials in standard form.

(a) $3x^2+5x^3+7-8x$ (b) $9x^4+2x-x^2+1$ (c) $3-2x-5x^2$

Polynomials are simply abstract representations of numbers that we see every day and they behave like these numbers as well. Let's look at adding polynomials together.

***Exercise* #5:** Consider the numbers 523 and 271.

(a) Write each as the sum of multiples of powers of 10 as previously done.

(b) Add these numbers by adding each individual power of 10.

(c) Use this idea to add:

$$\begin{array}{r} 5x^2+2x+3 \\ +2x^2+7x+1 \end{array}$$

(d) Find the sum of the polynomials $-4x^2+9x-3$ and $7x^2-5x+4$.

Finding sums of polynomials is fairly easy. Subtracting them, though, can lead to a lot of errors.

***Exercise* #6:** Find each of the following differences. Be careful and rewrite as an equivalent addition problem if necessary.

(a)
$$\begin{array}{r} 6x^2+5x+3 \\ -\ \underline{2x^2-4x+7} \end{array}$$

(b) $\left(4x^2-2x+7\right)-\left(-2x^2+x-3\right)$

***Exercise* #7:** For each of the following, write an equivalent polynomial in simplest standard form.

(a) $6x^2+2x-3-x^2+4x-1$ (b) $6x^2+2x-3-\left(x^2+4x-1\right)$

Name: ______________________________ Date: ______________

INTRODUCTION TO POLYNOMIALS

FLUENCY

1. Write each of the following integers as multiples of powers of 10. The first is done as a reminder of this process.

(a) 563 (b) 278 (c) 703

$$563 = 500 + 60 + 3$$
$$= 5 \cdot 100 + 6 \cdot 10 + 3$$
$$= 5 \cdot 10^2 + 5 \cdot 10 + 3$$

(d) 5,378 (f) 19,073

2. Consider the number 5,364.

(a) Write this number as the sum of multiples of powers of 10 as in #1.

(b) If $x = 10$, write an expression in terms of x for the number 5,364.

3. Which of the following would be the value of the expression $5x^3 + 2x^2 + 8x + 4$ when $x = 10$?

(1) 6,432 (3) 5,284

(2) 2,854 (4) 528

4. Which of the following would be the value of the expression $8x^3 + 2x + 3$ when $x = 10$?

(1) 823 (3) 8,203

(2) 8,023 (4) 8,230

5. Which of the following is *not* a polynomial expression?

(1) x^4 (3) $1 - 2x^3$

(2) 3^x (4) $6x + 1$

4

6. Write each of the following polynomial expressions in standard form.

(a) $7x^2+4x^3+5+2x$ (b) $4-x-5x^2$ (c) x^3+x-7x^2+2

(d) $2x+1-3x^3+5x^2$ (e) $4x^3-2x^2+6-8x$ (f) $y^5+y^{10}-y^2+y^7$

7. Find each of the following sums and differences. Write your answer in simplest standard form.

(a) $6x^2-2x+8+3x^2+7x-2$ (b) $x^3+4x^2-8x+3+x^3-x+1$

(c) $(5x^2+3x-1)-(3x^2-6x+4)$ (d) $(2x^3-5x^2+8x-1)-(-4x^3+8x^2-3x-9)$

(e) $4x^2+6x-3-3x^2+2x+4$ (f) $(4x^2+6x-3)-(3x^2+2x+4)$

APPLICATIONS

8. A box has a width that is 2 inches greater than its height and a length that is 6 inches greater than its height. It's volume is given by the polynomial expression x^3+8x^2+12x, where x is the box's height. What is the box's volume, in cubic inches, if its height is 10 inches?

(1) 1,812 (3) 182

(2) 1,920 (4) 2,180

REASONING

9. Polynomial expressions act a lot like integers because the structure of polynomials is based on the structure of integers. Based on the statement below about integers, make a statement about polynomials.

Statement About Integers: An integer added to an integer gives an integer.

Statement About Polynomials: ______________________________

____

5

Name: ______________________________ Date: ______________

MULTIPLYING POLYNOMIALS

Polynomials, as we saw in the last lesson, behave a lot like integers (whole numbers including the negatives). We saw that just like integers, **adding one polynomial to another polynomial results in a third polynomial.** The same will occur with multiplying them. First, a review problem.

***Exercise* #1:** **Monomials** are the simplest of **polynomials**. They consists of one **term** (terms are separated by addition and subtraction). Find the following products of monomials.

(a) $5x^3 \cdot 2x^2$

(b) $-3x \cdot -8x$

(c) $\frac{1}{2}x^2y^5 \cdot \frac{3}{4}x^9y$

We have also used the **Distributive Property** in previous lessons to multiply polynomials that are more complicated.

***Exercise* #2:** Find each of the following products in simplest form by using the distributive property once or twice.

(a) $2x(3x-1)$

(b) $x^2(4x^2+3)$

(c) $-2x^2y^3(2xy-5x)$

(d) $(x+2)(x-6)$

(e) $(2x+7)(x+3)$

(f) $(3x-2)(5x-1)$

Never forget that as we do these manipulations we are using **properties of equality** to produce **equivalent expressions.**

***Exercise* #3:** Consider the product of the two **binomial polynomials** $(x-1)(x-3)$.

(a) Find this product and express it as a **trinomial polynomial** written in standard form. Fill in the result in the first row (third column) of table (b).

(b) Fill out the table below using **TABLES** on your calculator to show they are equivalent.

x	$(x-1)(x-3)$	
0		
1		
2		
3		
4		

6

We can evaluate more complicated products, just as we have done in the past with normal numbers. The key will always be the careful use of the **distributive property**.

***Exercise* #4:** Find each of the following more challenging products.

(a) $(2x+5)^2$

(b) $(x+2)(x^2+4x+3)$

(c) $(x-4)(x+3)(x-5)$

(d) $(3x+2)^3$

***Exercise* #5:** Consider the product $(3x+2)(2x+1)$.

(a) Write this product as an equivalent trinomial expression in standard form.

(b) How can you use your answer from (a) to evaluate the product $(32)(21)$? Find the product and check using your calculator.

Name: ______________________________ Date: ______________

MULTIPLYING POLYNOMIALS

FLUENCY

1. Write the following products as polynomials in either x or t. The first is done as an example for you.

(a) $5x(2x-4)$

$=(5x)(2x)-(5x)(4)$

$=(5\cdot 2)(x\cdot x)-(5\cdot 4)(x)$

$=10x^2-20x$

(b) $3t(t+7)$

(c) $-4x(5x+1)$

(d) $4(t^2-5t+2)$

(e) $x(x^2-2x-3)$

(f) $-5t(2t^2+3t-7)$

2. Perhaps the most important type of polynomial multiplication is that of two binomials. Make sure you are **fluent** with this skill. Write each of the following **products** as an **equivalent polynomial** written in **standard form**. The first problem is done as an example using **repeated distribution.**

(a) $(x+5)(x-3)$

$=(x+5)(x)+(x+5)(-3)$

$=(x)(x)+(5)(x)+(x)(-3)+(-5)(3)$

$=x^2+5x-3x-15$

$=x^2+2x-15$

(b) $(x-10)(x-4)$

(c) $(x+3)(x+12)$

(d) $(2x+3)(5x+8)$

(e) $(4x-1)(x+2)$

(f) $(6x-5)(4x-3)$

3. Never forget that squaring a binomial also a process of repeated distribution. Write each of the following perfect squares as **trinomials in standard form.**

(a) $(x+3)^2$

(b) $(x-10)^2$

(c) $(2t+3)^2$

4. An interesting thing happens when you multiply two **conjugate binomials**. Conjugates have the property of having the same **terms** but differ by the operation between the two terms (in one case addition and in one case subtraction). Multiply each of the following **conjugate pairs** and state your answers in **standard form**. The first is done as an example

(a) $(x+3)(x-3)$

$=x(x-3)+3(x-3)$

$=x^2-3x+3x-9$

$=x^2-9$

(b) $(x-5)(x+5)$

(c) $(10+x)(10-x)$

(d) $(2t+3)(2t-3)$

(e) $(5t+1)(5t-1)$

(f) $(8-3t)(8+3t)$

5. Write each of the following products in standard polynomial form.

(a) $(x+3)(x-2)(x-8)$

(b) $(x+2)(x-2)(x+3)(x-3)$ (Hint: try to use #4)

REASONING

6. Notice again how similar polynomials are to integers, i.e. the set $\{...-3,-2,-1,0,1,2,3...\}$. Write a statement below for polynomials based on the statement about integers.

Statement About Integers: An integer times an integer produces an integer.

Statement About Polynomials: ____

____

7. Consider the product $(3x+1)^2$.

(a) Write this product in standard trinomial form.

(b) Use your answer in part (a) to determine the value of 31^2 without your calculator.

Name_______________________

Date______________ Period____

Polynomials

Simplify each expression.

1) $(3n^4 + 5 - 3n^3) - (2n^4 - 5n^3 - 1)$

2) $(8m^4 - 6m + 7m^2) + (3m^4 + 2m^2 + 2m)$

3) $(3n + 1 - 3n^2) - (5 - 7n - 4n^2)$

4) $(2a^3 - 4a^3b + 7ab) - (5a^3b + 8ab + 6a^3)$

5) $(7ab^2 - a^2b - 7a) + (5a - 5ab^2 - 8a^2b)$

6) $(6n^3 - 1 + 6m^4n^3) + (7m^4n^3 + 3n^3 - 7m^4n^4)$

7) $(4n^3 - 2mn^4 - 8n^2) - (5n^2 - 6mn^4 - 2n^4)$

Find each product.

8) $5p(7p - 8)$

9) $8(2n - 3)$

10) $7x^2(6x^2 - x - 1)$

11) $2(8x^2 + 3x + 4)$

12) $(-8x - 4)(6x - 5)$

13) $(-8a + 1)(2a - 3)$

10

14) $(7x-2)(-8x+3)$

15) $(-7x+5)(7x+3)$

16) $(-3r-7)(-7r+8)$

17) $(-8n-5)(-3n+3)$

18) $(4x-2)(-7x+2)$

19) $(-5b+1)(b-8)$

20) $(-7x-8)(2x-4)$

21) $(b-4)(-3b+1)$

22) $(4n-1)(2n-8)$

23) $(3a+7)(8a-8)$

24) $(7v-8)(-7v-6)$

25) $(-4m-8)(6m+8)$

26) $(-n+1)(-4n+8)$

27) $(4x-7)(3x^2+2x+4)$

28) $(2p+4)(5p^2-p-2)$

29) $(2x+3)(2x^2+7x-4)$

Name: ______________________________ Date: ______________

INTRODUCTION TO QUADRATIC FUNCTIONS

We have now studied **linear** and **exponential** functions. These functions were relatively simple because they were either **always increasing** or **always decreasing** for their entire **domains.** We now will start to study other functions, most notably **quadratic functions**, which are a type of **polynomial function**. Their definition is shown below:

QUADRATIC FUNCTIONS

Any function that can be placed in the form: $y = ax^2 + bx + c$, where $a \neq 0$, but b and c can be zero.

***Exercise* #1:** Read the definition above for quadratic functions and answer the following questions.

(a) Why is it important for the **leading coefficient** to be nonzero?

(b) Circle the choices below that are quadratic functions.

$y = x^2 - 3$ $\qquad$ $y = x^3 + 2x^2 - 4$

$y = x^2 + \sqrt{x} + 7$ $\qquad$ $y = 10 - x^2$

(c) Given the quadratic function $y = 10 - 3x^2 + 7x$, write it in standard form and state the value of the leading coefficient.

(d) If $f(x) = 2x^2 - 3x + 1$, then find, without using your calculator, the value of $f(-2)$. What point must lie on this quadratic's graph based on this calculation?

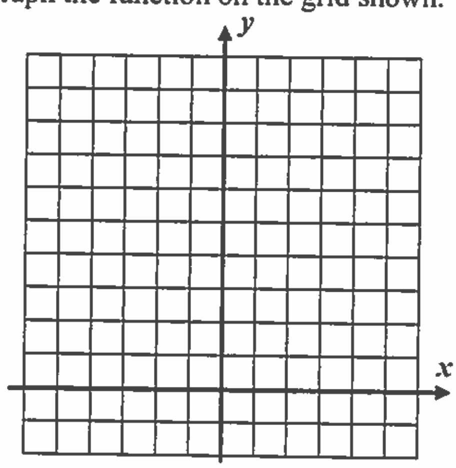
Quadratics still behave in similar ways to other functions. Inputs go in, outputs come out. But, they start to behave differently from **linear** and **exponential** functions because sometimes **outputs repeat** for quadratics.

***Exercise* #2:** Consider the simplest of all quadratic functions, $f(x) = x^2$.

(a) Fill out the table below without using your calculator.

x	-3	-2	-1	0	1	2	3
$y = x^2$							

(b) Graph the function on the grid shown.

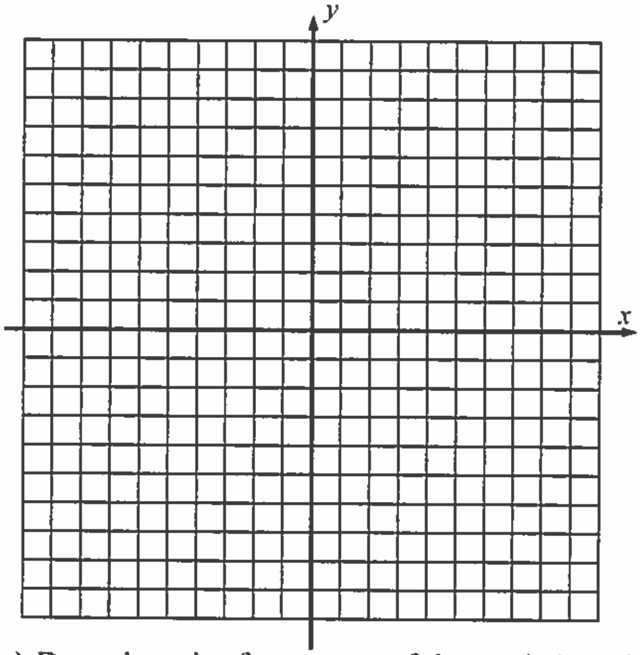


(c) What is the **range** of this quadratic function?

Quadratic functions can obviously be more complicated than our last example, but, strangely enough, they all have the same general shape, which is known as a **parabola**. Let's explore the next quadratic function with the help of technology. We will also introduce some important terminology.

***Exercise* #3:** Consider the quadratic function $y = x^2 - 2x - 8$.

(a) Using your calculator to help generate a table, graph this parabola on the grid given. Show a table of values that you use to create the plot.



(b) State the **range** of this function.

(c) Over what **domain interval** is the function **increasing**?

(d) State the coordinates of the parabola's turning point (also known as its vertex and its minimum point).

(e) Draw the axis of symmetry of the parabola and write its equation below and on the graph.

(f) What are the x-intercepts of this function? These are also known as the function's **zeroes**. Why does this name make sense? As a suggestion, write out their full xy-pair coordinates.

***Exercise* #4:** The quadratic function $f(x)$ has selected values shown in the table below.

(a) What are the coordinates of the turning point?

(b) What is the range of the quadratic function?

x	$f(x)$
-1	4
0	9
1	12
2	13
3	12
4	9
5	4
6	-2

13

Name: ______________________ Date: ____________

Introduction to Quadratic Functions
Common Core Algebra I Homework

Fluency

1. Which of the following is a quadratic function?

(1) $y=3x-2$ (3) $y=x^2-4$

(2) $y=x^3+2x^2-1$ (4) $y=6(2)^x$

2. The quadratic function $y=9-x^2+4x$ written in standard form would be

(1) $y=-x^2+4x+9$ (3) $y=x^2-4x+9$

(2) $y=x^2-9x+4$ (4) $y=-x^2-4x+9$

3. Which of the following would be the leading coefficient of $f(x)=6-x+7x^2$?

(1) -1 (3) 7

(2) 6 (4) -7

4. Which of the following points lies on the graph of $y=x^2-5$?

(1) $(3,-2)$ (3) $(5,0)$

(2) $(-2,-1)$ (4) $(-1,-6)$

5. A quadratic function is partially given in the table below. Which of the following are the coordinates of its turning point?

(1) $(0,6)$ (3) $(3,15)$

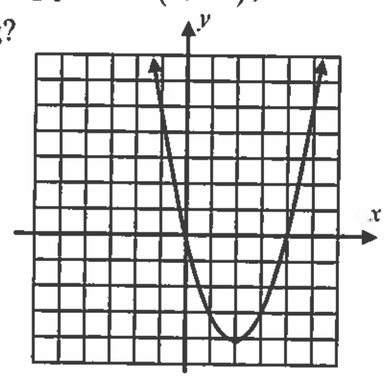
(2) $(10,2)$ (4) $(7,-1)$

x	-2	-1	0	1	2	3
y	10	7	6	7	10	15

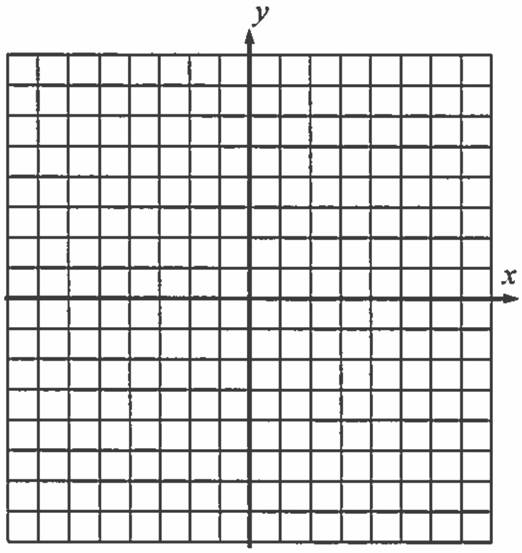
6. Given the quadratic function shown below whose turning point is $(2,-4)$, which of the following gives the domain interval over which this function is decreasing?

(1) $x>-4$ (3) $x>2$

(2) $x<-4$ (4) $x<2$



7. Consider the function $f(x) = x^2 + 2x - 3$.



(a) Using your calculator, create an accurate graph of $f(x)$ on the grid provided.

(b) State the coordinates of the turning point of $f(x)$. Is this point a maximum or minimum?

(c) State the range of this quadratic function.

(d) State the **zeroes** of this function (the x-intercepts).

(e) Over what interval is this function **negative**? In other words, over what x-values is the output (or y-value) to this function negative?

(f) Over what interval is this function **increasing**?

(g) Determine the average rate of change of this function over the interval $-2 \le x \le 4$.

Reasoning

8. A quadratic function $g(x)$ is shown partially in the table below. The turning point of the function has the coordinates $(3, -8)$. Think about how outputs repeat in a quadratic function and answer the following.

x	-1	0	1	2	3	4	5	6	7
$g(x)$	24		0	-6	-8			10	

(a) Fill in the missing output values from the table.

(b) What are the zeroes of the function?

(c) What is this function's y-intercept?

(d) For the domain interval $-1 \le x \le 7$, what is the range of the function?

Name: ______________________________ Date: ______________

SOLVING QUADRATICS BY INVERSE OPERATIONS

Now that we have a good feeling for square roots, we can use them to help us solve special types of quadratic equations (those equations involving a squared quantity). Let's make sure we first understand a basic concept.

***Exercise* #1:** Solve each of the following equations for all values of x. Write your answers in simplest radical form.

(a) $x^2 = 16$

(b) $x^2 = 100$

(c) $x^2 = 20$

So, the key here is that the **inverse operation** to **squaring** is taking a **square root**. BUT, when you do this, you always introduce both a positive and negative answer. Squaring is a **non-reversible** process, meaning that you can't simply undo it.

Now, let's add some additional operations. Recall that we always solve equations by undoing operations in the opposite order in which they have been done. And in terms of order of operations, exponents essentially come first, so they will be "undone" last.

***Exercise* #2:** Solve each of the following equations for all values of x by using inverse operations. In each case your final answers will be rational numbers.

(a) $2x^2 + 10 = 28$

(b) $\dfrac{x^2}{2} - 5 = 3$

(c) $(x-2)^2 = 25$

(d) $2(x+5)^2 - 50 = 150$

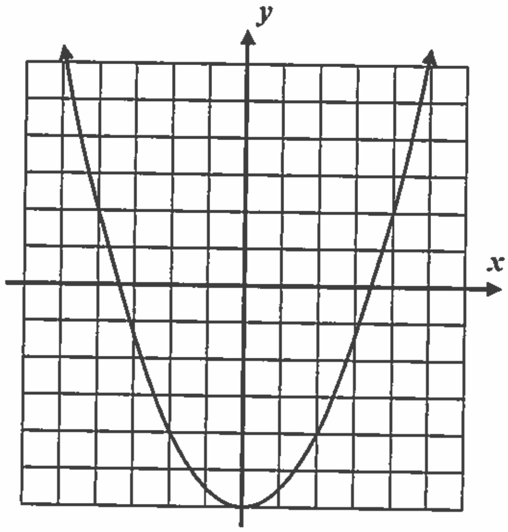
Of course, there is no reason our answers must come out as rational numbers as in Exercise #3. We can also have answers to these types of equations that involve **irrational numbers.** In these cases we are typically asked, for some unknown reason, to express our answers in **simplest radical form.**

***Exercise* #3:** Solve each of the following quadratic equations by using inverse operations. Express all of your answers in simplest radical form.

(a) $5x^2 - 2 = 38$

(b) $(x-3)^2 + 10 = 38$

***Exercise* #4:** Francis graphs the parabola $y = \frac{1}{2}x^2 - 6$ on the grid below. He believes that the quadratic has zeroes of -3.5 and 3.5.



(a) Find the zeroes of this function in simplest radical form and explain why Francis must be incorrect.

(b) Francis was incorrect based on (a), but not too far off? How can you tell how good his estimate was?

***Exercise* #5:** Find the zeroes of the function $f(x) = (x+4)^2 - 20$ in simplest radical form. Then, express them in terms of a decimal rounded to the nearest *hundredth*.

Name: ______________________________ Date: ________________

Solving Quadratics Using Inverse Operations

Fluency

1. Solve each of the following quadratics by applying inverse operations. In each case, your answers will be rational numbers. Always write them in simplest form.

(a) $2x^2=98$

(b) $(x+3)^2=25$

(c) $x^2-11=53$

(d) $\dfrac{(x-4)^2}{3}=12$

(e) $20(x+1)^2=5$

(f) $-2(x-7)^2+5=-195$

(g) $(2x+1)^2-6=19$

2. Which of the following is the solution set to the equation $\dfrac{(x-6)^2}{2}+4=36$?

(1) $\{0,12\}$

(3) $\{-2,16\}$

(2) $\{-4,8\}$

(4) $\{-2,14\}$

18

3. Solve each of the following quadratic equations by using inverse operations. Express each of your answers simplest radical form.

(a) $\frac{1}{2}x^2-4=0$

(b) $(x-5)^2=18$

(c) $2x^2+7=71$

(d) $5(x+2)^2+37=487$

(e) $\frac{(x-4)^2}{3}+8=17$

APPLICATIONS

4. The height, h, of an object above the ground in feet, can be modeled as a function of time, t, in seconds using the equation:

$$h(t)=-16(t-2)^2+400\text{, for } t\geq 0$$

(a) Find the time, in seconds, when the object reaches the ground, $h=0$.

(b) Find all time(s) when the object is at a height of 200 feet. Round your answer(s) to the nearest tenth of a second.

5. Which of the following choices represent the zeroes of the function $g(x)=2(x-5)^2-400$?

(1) $x=5\pm 10\sqrt{2}$

(3) $x=-5\pm 20\sqrt{5}$

(2) $x=-10\pm 20\sqrt{5}$

(4) $x=5\pm 2\sqrt{10}$

Use the Quadratic Formula to Solve Quadratic Equations

Example 1 Solve the equation $x^2+3x-4=0$ by using the quadratic formula

Solution $$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad for \quad ax^2+bx+c=0$$

Plug in the values for a, b, and c into the quadratic formula above, in order to solve for x. In our case a = 1, b = 3, and c = -4.

$$x_{1,2}=\frac{-3\pm\sqrt{3^2-4(1)(-4)}}{2(1)}=\frac{-3\pm\sqrt{9+16}}{2}=\frac{-3\pm\sqrt{25}}{2}=\frac{-3\pm5}{2}$$

$$x_1=1 \quad x_2=-4$$

PROBLEMS Solve the equation by using the quadratic formula.

1. $x^2+7x+12=0$

2. $x^2-8x+12=0$

3. $x^2-x-2=0$

4. $-2x^2+4x+8=0$

Example 2 Solve the equation $x^2+3x+10=0$ by using the quadratic formula

Solution $x_{1,2}=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ *for* $ax^2+bx+c=0$

Plug in the values for a, b, and c into the quadratic formula above, in order to solve for x. In our case a = 1, b = 3, and c = 10.

$$x_{1,2}=\frac{-3\pm\sqrt{3^2-4(1)(10)}}{2(1)}=\frac{-3\pm\sqrt{9-40}}{2}=\frac{-3\pm\sqrt{-31}}{2}=\frac{-3\pm\sqrt{31}\cdot\sqrt{-1}}{2}=\frac{-3\pm\sqrt{31}i}{2}$$

$$x_1=-\frac{3}{2}+\frac{\sqrt{31}i}{2} \qquad x_2=-\frac{3}{2}-\frac{\sqrt{31}i}{2}$$

PROBLEMS Solve the equation by using the quadratic formula.

5. $x^2+4x+8=0$

6. $-3x^2+x-2=0$

How do you know whether you end up with solution featuring 2 real numbers, 1 real number, or 2 imaginary numbers?

Answer: You solve for the discriminant. That is the number under the square root in the expression $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The discriminant is $b^2 - 4ac$.

If the discriminant is **positive:** 2 real number solutions

If the discriminant is **negative:** 2 imaginary number solution

If the discriminant is **zero:** 1 real number solution

PROBLEMS Solve for the discriminant to decide whether the equation has 2 real solutions, 2 imaginary solutions or 1 real solution.

5. $2x^2 - 3x + 1 = 0$

6. $x^2 + 4x + 4 = 0$

7. $-2x^2 + 4x - 8 = 0$

8. $-1.5x^2 + 3x + 2 = 0$

23

What if the quadratic equation isn't equal to zero?

Solve each equation with the quadratic formula.

1) $5x^2 - 119 = 6$

2) $n^2 + 12n + 11 = -9$

3) $6x^2 - 17 = 7$

4) $3m^2 + m - 13 = -9$

5) $6m^2 + 5m - 66 = -10$

6) $6k^2 - k = 1$

24 7) $9k^2 + 12k - 10 = 5$

8) $3p^2 - 27 = 0$

9) $a^2 - 30 = a$

10) $2v^2 + 11v = -11$

11) $6x^2 + 4 = 10x$

12) $9x^2 = 5x + 20$

13) $n^2 + 32 = 12n$

14) $-5n^2 + 2 = -6 - 11n^2$

Name: ______________________________ Date: ______________

COMPLETING THE SQUARE

The turning point of a parabola and its general shape are relatively easy to determine if the quadratic function is written in its **shifted or vertex form**. Review this in the first exercise.

***Exercise* #1:** Given the function $y=(x-3)^2+2$ do the following.

(a) Give the coordinates of the turning point.

(b) Determine the range by drawing a rough sketch.

The question then is how we take a quadratic of the form $y=ax^2+bx+c$ and put it into its shifted form. This procedure is known as **Completing the Square**. But, it needs some additional review.

***Exercise* #2:** Write each of the following as an equivalent trinomial.

(a) $(x+5)^2$

(b) $(x-1)^2$

(c) $(x+4)^2$

***Exercise* #3:** Given each trinomial in Exercise #2 of the form x^2+bx+c, what is true about the relationship between the value of b and the value of c? Illustrate.

***Exercise* #4:** Each of the following trinomials is a perfect square. Write it in factored (or perfect square) form.

(a) $x^2+20x+100$

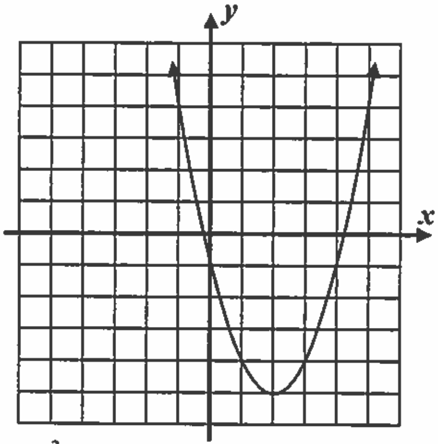
(b) x^2-6x+9

(c) x^2+2x+1

26

We are finally ready to learn the method of **Completing the Square**. This method has many uses, but the one we will work with today is to manipulate equations of quadratics from their **standard form** to their **vertex form**.

***Exercise* #5:** The quadratic $y=x^2-4x-1$ is shown graphed below.



(a) Consider only the binomial x^2-4x. What would you need to add on to it to create a perfect square trinomial? (See Exercise #3).

(b) In order to add zero to the binomial x^2-4x, what should we subtract to offset adding 4 to make it a perfect square?

(c) Use the Method of Competing the Square now to rewrite the trinomial x^2-4x-1 in an equivalent, shifted form. According to this form, what are the coordinates of the vertex? Verify by examining the graph.

This procedure is what is known as an **algorithm**. In other words, we follow a recipe. Here it is:

COMPLETING THE SQUARE

For the quadratic $y=x^2+bx+c$ (note that $a=1$).

1. Find half of the value of b, i.e. $\frac{b}{2}$ 2. Square it, i.e. $\left(\frac{b}{2}\right)^2$ 3. Add and subtract it

There is nothing like practice on these.

***Exercise* #6:** Write each quadratic in vertex form by Completing the Square. Then, identify the quadratic's turning point. The last two problems will involve fractions. Stick with it!

(a) $y=x^2+6x-2$ (b) $y=x^2-2x+11$ (c) $y=x^2-10x+27$

(d) $y=x^2+8x$ (e) $y=x^2+5x+4$ (f) $y=x^2-9x-2$

Name: ____________________ Date: ____________

COMPLETING THE SQUARE

FLUENCY

1. Find each of the following products in standard form.

(a) $(x+4)^2$ (b) $(x-1)^2$ (c) $(x+8)^2$

(d) $(x-7)^2$ (e) $(x+2)^2$ (f) $(x-10)^2$

2. Each of the following trinomials is a perfect square. Write it in factored form, i.e. $(x+a)^2$ or $(x-a)^2$.

(a) x^2+6x+9 (b) $x^2-22x+121$ (c) $x^2+10x+25$

(d) $x^2+30x+225$ (e) x^2-2x+1 (f) $x^2-18x+81$

3. Place each of the following quadratic functions, written in standard form, into vertex form by completing the square. Then, identify the coordinates of its turning point.

(a) $y=x^2-12x+40$ (b) $y=x^2+4x+14$ (c) $y=x^2-24x+146$

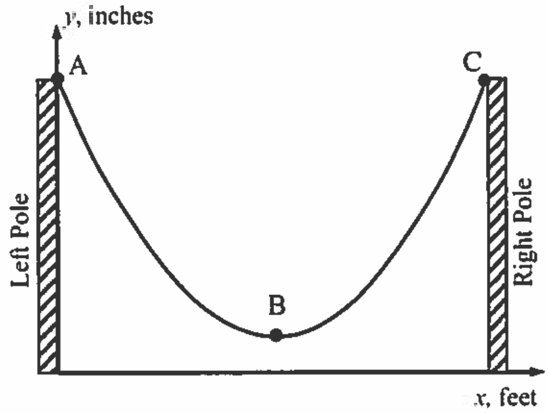
28

APPLICATIONS

4. A cable is attached at the same height from two poles and hangs between them such that its height above the ground, y, in inches, can be modeled using the equation:

$$y = x^2 - 16x + 67$$

where x represents the horizontal distance from the left pole, in feet.



(a) What height is point A above the ground? Show your work and use proper units.

(b) Write the equation in vertex form.

(c) What is the difference in the heights of points A and B? Show your analysis and include units.

(d) What is the horizontal distance that separates points A and C? Explain your reasoning.

REASONING

5. Use the vertex form of each of the following quadratic functions to determine which has the lowest y-value.

$y = x^2 - 8x + 6$ $\qquad\qquad$ $y = x^2 + 6x + 1$

6. Two quadratic functions are shown below, $f(x)$ and $g(x)$. Determine which has the lower minimum value. Explain how you arrived at your answer.

$f(x) = x^2 + 10x$

x	3	4	5	6	7	8	9
$g(x)$	−9	−14	−17	−18	−17	−14	−9

Completing the Square

Goal: Solve quadratic equations by completing the square.

To complete the square for the expression $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$ on both sides of the equation. Once you completed the square your expression looks like: $x^2 + bx + (\frac{b}{2})^2$

The factored form after completion of the square is $(x + \frac{b}{2})^2$

Example 1 Find the value of c that makes the expression a perfect square: $x^2 - 10x + c$

Solution:	Step 1:	-10	Find the coefficient of x
	Step 2:	$\frac{-10}{2} = -5$	Half the coefficient of x
	Step 3:	$(-5)^2 = 25$	Square the result from step 2
	Step 4:	$x^2 - 10x + 25$	Replace c with the result from step 3
	Step 5:	$(x-5)^2$	Write in factored form

PROBLEMS Find the value of c that makes the expression a perfect square. Then write the expression in factored form.

1. $x^2 + 12x + c$
2. $x^2 - 18x + c$
3. $x^2 - 4x + c$
4. $x^2 + 36x + c$
5. $x^2 - 8x + c$
6. $x^2 - 20x + c$

Example 2 Solving $ax^2+bx+c=0$ *when* $a=1$.

Solve $x^2-16x+8=0$ by completing the square.

Solution:

Step 1: $x^2-16x+8=0$ Write the original equation.

Step 2: $x^2-16x=-8$ Write the left side in the form of x^2+bx.

Step 3: $x^2-16x+64=-8+64$ Add $\left(\frac{16}{2}\right)^2$ on both sides to complete the square.

Step 4: $(x-8)^2=56$ Write the left side in factored form.

Step 5: $\sqrt{(x-8)^2}=\pm\sqrt{56}$ Take the square root on both sides.

Step 6: $(x-8)=\pm\sqrt{56}$ The square root eliminates the square on the left side.

Step 7: $x=8\pm2\sqrt{14}$ Simplify: $\sqrt{56}=\sqrt{4}\cdot\sqrt{14}=2\cdot\sqrt{14}$.

Therefore, the solutions are $8+2\sqrt{14}$ *and* $8-2\sqrt{14}$.

PROBLEMS Solve the equation by completing the square.

1. $x^2-10x+6=0$

2. $x^2+16x-9=0$

3. $x^2-18x-5=0$

4. $x^2+6x+12=0$

Example 2 Solving $ax^2+bx+c=0$ *when* $a\neq 1$.

Solve $2x^2-32x+16=0$ **by completing the square.**

Solution: Step 1: $2x^2-32x+16=0$ Write the original equation.

Step 2: $x^2-16x+8=0$ Divide both sides by 2.

Proceed as in example 1 to arrive at the solutions $8+2\sqrt{14}$ *and* $8-2\sqrt{14}$.

PROBLEMS Solve the equation by completing the square (Hint: First divide by the coefficient of x^2).

5. $5x^2-10x+30=0$

6. $4x^2+8x-36=0$

7. $5x^2-10x+30=0$

8. $2x^2-4x-10=0$

9. $3x^2-9x-27=0$

10. $8x^2-24x-32=0$

Name: ______________________________ Date: ______________

FACTORING POLYNOMIALS

Factoring expressions is one of the **gateway skills** that is necessary for much of what we do in algebra for the rest of the course. The word **factor** has two meanings and both are important.

THE TWO MEANINGS OF FACTOR

1. **Factor (verb):** To rewrite an algebraic expression as an **equivalent product**.
2. **Factor (noun):** An algebraic expression that is one part of a larger factored expression.

***Exercise* #1:** Consider the expression $6x^2+15x$.

(a) Write the individual terms $6x^2$ and $15x$ as completely factored expressions. Determine their **greatest common factor**.

$6x^2 =$ $\qquad$ $15x =$

(b) Using the Distributive Property, rewrite $6x^2+15x$ as a product involving the **gcf** from (a).

(c) Evaluate both $6x^2+15x$ and the factored expression you wrote in (b) for $x=2$. What do you find? What does this support about the two expressions?

It is important that you are **fluent** reversing the **distributive property** in order to factor out a common factor (most often the greatest common factor). Let's get some practice in the next exercise just identifying the greatest common factors.

***Exercise* #2:** For each of the following sets of monomials, identify the greatest common factor of each. Write each term as an extended product (if necessary).

(a) $12x^3$ and $18x$

(b) $5x^4$ and $25x^2$

(c) $21x^2y^5$ and $14xy^7$

(d) $24x^3, 16x^2$, and $8x$

(e) $20x^3, -12x^2$, and $28x$

(f) $18x^2y^2, 45x^2y$, and $90xy^2$

Once you can identify the greatest common factor of a set of monomials, you can then easily use it and the distributive property to write equivalent factored expressions.

***Exercise* #3:** Write each polynomial below as a factored expression involving the greatest common factor of the polynomial.

(a) $6x^2+10x$

(b) $3x-24$

(c) $10x^2-15x$

(d) $4x^2+8x+24$

(e) $6x^3-8x^2+2x$

(f) $10x^3-35x^2$

(g) $10x^2-40x-50$

(h) $8x^4-2x^2$

(i) $8x^3+24x^2-32x$

Being able to **fluently** factor out a gcf is an essential skill. Sometimes greatest common factors are more complicated than simple monomials. We have done this type of factoring back in Unit #1.

***Exercise* #4:** Rewrite each of the following expressions as the product of two binomials by factoring out a common binomial factor.

(a) $(x+5)(x-1)+(x+5)(2x-3)$

(b) $(2x-1)(2x+7)-(2x-1)(x-3)$

Name: ______________________________ Date: ________________

Factoring Polynomials

Fluency

1. Identify the greatest common factor for each of the following sets of monomials.

(a) $6x^2$ and $24x^3$

(b) $15x$ and $10x^2$

(c) $2x^4$ and $10x^2$

(d) $2x^3, 6x^2,$ and $12x$

(e) $16t^2, 48t,$ and 80

(f) $8t^5, 12t^3,$ and $16t$

2. Which of the following is the greatest common factor of the terms $36x^2y^4$ and $24xy^7$?

(1) $12xy^4$

(3) $6x^2y^3$

(2) $24x^2y^7$

(4) $3xy$

3. Write each of the following as equivalent products of the polynomial's greatest common factor with another polynomial (of the same number of terms). The first is done as an example.

(a) $8x-28$

$=4(2x-7)$

(b) $50x+30$

(c) $24x^2+32x$

(d) $18-12x$

(e) $6x^3+12x^2-3x$

(f) x^2-x

(g) $10x^2+35x-20$

(h) $21x^3-14x$

(i) $36x-8x^2$

(j) $30x^3-75x^2$

(k) $-16t^2+96t$

(l) $4t^3-32t^2+12t$

4. Which of the following is *not* a correct **factorization** of the binomial $10x^2+40x$?

(1) $10x(x+4)$

(3) $5x(2x+4)$

(2) $10(x^2+4x)$

(4) $5x(2x+8)$

5. Rewrite each of the following expressions as the product of two binomials by factoring out a common binomial factor. Watch out for the subtraction problems (b) and (d).

(a) $(x+5)(x+1)+(x+5)(x+8)$

(b) $(2x-1)(3x+5)-(2x-1)(x+4)$

(c) $(x-7)(x-9)+(x-7)(4x+5)$

(d) $(x+1)(5x-7)-(x+1)(x-3)$

APPLICATIONS

6. The area of a rectangle is represented by the polynomial $16x^2+56x$. The width of the rectangle is given by the binomial $2x+7$.

(a) Give a monomial expression in terms of x for the length of the rectangle. Show how you arrived at your answer.

(b) If the length of the rectangle is 80, what is the width of the rectangle? Explain your thinking.

REASONING

7. These crazy polynomials keep acting like integers. We can factor integers to determine their factors. We can also do the same for polynomials.

(a) List all of the positive factors of the integer 12 by writing all possible positive integer products (such as $12=3\cdot 4$).

(b) List all of the factors of $2x^2-6x$ by also writing all possible products, such as $2(x^2-3x)$.

8. Which of the following is *not* a factor of $4x^2+12x$?

(1) $x+3$

(2) x

(3) $3x$

(4) 4

Solve Polynomial Equations in Factored Form

Goal: Solve polynomial equations.

Example 1 Solve (x-2)(x+5)=0

Solution: $(x-2)(x+5)=0$

$(x-2)=0$ or $(x+5)=0$

$x=2$ or $x=-5$

PROBLEMS

1. $(x-5)(x+3)=0$

2. $(x+2)(x+1)=0$

3. $(2x-4)(x-1)=0$

4. $(p-7)(p+8)=0$

5. $(2x+8)(x+10)=0$

6. $(b+3)(b+2)=0$

7. $(2x-6)(x+\frac{1}{2})=0$

8. $(b-\frac{3}{2})(2b+3)=0$

Factor $x^2 + bx + c$

Goal Factor trinomials in the form of $x^2 + bx + c$

Use the following information to factor trinomials of the form $x^2 + bx + c$:

$x^2 + bx + c = (x + p)(x + q)$ where : $p + q = b$ and $pq = c$

Example 1 Factor $x^2 + 7x + 12$

Solution We need to find two numbers whose sum is 7 and whose product is 12.

In other words: Find two numbers which when you add them (sum) give you 7 and when you multiply them (product) give you 12.

There are three pairs of numbers whose product is 12:

12 x 1 6 x 2 4 x 3

Only one pair gives you a sum of 7: $4 + 3 = 7$

Therefore, we can factor $x^2 + 7x + 12$ as $(x + 4)(x + 3)$

Check: $(x + 4)(x + 3) = x^2 + 3x + 4x + 12 = x^2 + 7x + 12$

PROBLEMS

1. $x^2+7x+10$

2. $x^2+9x+20$

3. $x^2+7x+12$

4. $x^2+8x+15$

5. $x^2+10x+21$

6. $x^2+15x+56$

7. x^2+4x+3

8. $x^2+11x+30$

Example 2 Factor $x^2 - 1x - 12$

Solution We need to find two numbers whose sum is -1 and whose product is -12.

In other words: Find two numbers which when you add them (sum) give you -1 and when you multiply them (product) give you -12.

There are six pairs of numbers whose product is -12:

12 x (-1)	6 x (-2)	4 x (-3)
(-12) x 1	(-6) x 2	(-4) x 3

Only one pair gives you a sum of -1: -4 + 3 = 1

Therefore, we can factor $x^2 - 1x - 12$ as

$(x - 4)(x + 3)$

Check: $(x - 4)(x + 3) = x^2 + 3x - 4x - 12 = x^2 - 1x - 12$

PROBLEMS

Factor

7. $x^2 + 2x - 15$

8. $x^2 + x - 12$

9. $x^2 + 2x - 8$

10. $x^2 - 1x - 2$

11. $x^2 - 2x - 24$

12. $x^2 - 5x - 24$

Example 3 Factor $x^2 - 7x + 12$

Solution We need to find two numbers whose sum is -7 and whose product is +12.

In other words: Find two numbers which when you add them (sum) give you -7 and when you multiply them (product) give you +12.

The only way to find such a pair is by multiplying two negative numbers.

There are three pairs of negative numbers whose product is +12:

(-12) x (-1) (-6) x (-2) (-4) x (-3)

Only one pair gives you a sum of -7: -4 - 3 = -7

Therefore, we can factor $x^2 - 7x + 12$ as

$(x - 4)(x - 3)$

Check: $(x - 4)(x - 3) = x^2 - 3x - 4x + 12 = x^2 - 7x + 12$

PROBLEMS

Factor

13. $x^2 - 8x + 15$

14. $x^2 - 3x + 2$

15. $x^2 - 9x + 20$

16. $x^2 - 11x + 30$

18. $x^2 - 10x + 24$

19. $x^2 - 11x + 24$

Example 4	Solve the equation $x^2 - x = 12$	
Solution	$x^2 - x = 12$	Write the original equation
	$x^2 - x - 12 = 0$	Subtract 12 from both sides
	$(x - 4)(x + 3) = 0$	Factor the left side
	$x - 4 = 0$ or $x = 4$	Solve
	$x + 3 = 0$ or $x = -3$	
	The solutions are 4 and -3.	

PROBLEMS

Solve the equation

20. $x^2 - 9x = -20$

21. $x^2 - 9x - 36 = 0$

22. $x^2 - 13x + 42 = 0$

23. $x^2 + 11x + 24 = 0$

Find the zeros of the polynomial function (Hint: set f(x)=0)

24. $f(x) = x^2 - 5x - 36$

25. $f(x) = x^2 + 8x - 20$

42

Factor $ax^2 + bx + c$

Goal	Factor trinomials in the form of $ax^2 + bx + c$

Example 1 Factor $4x^2 + 10x + 6$

Solution In order to solve these types of trinomials we need to use some trial and error. That means we make an educated guess and see whether it works.

First we only look at the factors $4x^2$ and 6.
We have two choices to factor $4x^2$: $(2x)(2x)$ or $(4x)(1x)$
We have also two choices to factor 6: (2)(3) or (6)(1)

Lets start with: $(2x+6)(2x+1)$.
Now multiply the "inners" $(6)(2x)=12x$
Then multiply the "outers": $(2x)(1)=2x$
Add them up: $12x + 2x = 14x$
$14x$ is not $10x$, therefore we have to try another combination.

Try $(2x+2)(2x+3)$
Now multiply the "inners" $(2)(2x)=4x$
Then multiply the "outers": $(2x)(3)=6x$
Add them up: $4x + 6x = 10x$
$10x$ is the correct value, therefore
$4x^2 + 10x + 6 = (2x+2)(2x+3)$

PROBLEMS

Factor the trinomial

1. $3x^2+10x+8$

2. $4x^2+6x+2$

3. $6x^2+23x+20$

4. $10x^2+19x+6$

Factor Special Products

Goal	Factor special products

Difference of Two Squares Pattern

$a^2-b^2=(a+b)(a-b)$ Example: $9x^2 - 16 = (3x)^2 - 4^2 = (3x+4)(3x-4)$

Example 1 Factor the polynomial

a. $x^2-4 = (x+2)(x-2)$

b. $4x^2-1 = (2x+1)(2x-1)$

c. $20-125x^2 = 5(4-25x^2) = 5(2+5x)(2-5x)$

<u>PROBLEMS</u>

Factor the polynomial

1. x^2-121
2. $9n^2-64$
3. $8x^2-2$

Perfect Square Trinomial Pattern

$a^2 + 2ab + b^2 = (a + b)^2$

Example: $x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2 = (x + 3)^2$

Perfect Square Trinomial Pattern

$a^2 - 2ab + b^2 = (a - b)^2$

Example: $x^2 - 6x + 9 = x^2 - 2(x)(3) + 3^2 = (x - 3)^2$

<u>PROBLEMS</u>

Factor the polynomial

4. $x^2+12x+36$
5. $n^2+10n+25$
6. $x^2+14x+49$
7. $x^2-12x+36$
8. x^2-4x+4
9. x^2-2x+1

Example 3 Solve a polynomial equation

Solve the equation $x^2 - 100 = 0$

Solution $x^2 - 100 = 0$ Write the original equation

$x^2 - 10^2 = 0$ Write left side as a^2-b^2

$(x+10)(x-10) = 0$ Difference of two squares pattern

$(x+10)=0$ or $(x-10) = 0$

$x = -10$ or $x = 10$

PROBLEMS

10. $x^2 - 12x + 36 = 0$

11. $x^2 - 4 = 0$

12. $x^2 + 14x + 49 = 0$

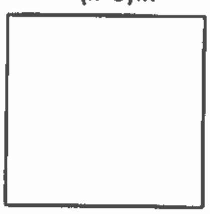
13. $x^2 - 9 = 0$

14. $x^2 + 8x + 15 = 0$

15. $x^2 - 2x + 1 = 0$

16. Find the value of x in the square.

(x+3)in



(x+3)in

Area = 16 in^2

Factor Polynomials Completely

Goal Factor polynomials completely

Try to group terms and look for common factors.

Example 1 Factor out common binomial

a. Factor the expression $5x^2(x-3)-4(x-3)$

Solution $5x^2(x-3)-4(x-3) = (5x^2-4)(x-3)$

b. Factor the expression $3x^2(x-3)+24(3-x)$

Solution $3x^2(x-3) + 24(3-x) = 3x^2(x-3) - 24(x-3) = (3x^2-24)(x-3)$

PROBLEMS

1. $11x(x+7)+3(x+7)$
2. $x^2(x+1)-4(x+1)$
3. $x^2(14x+9)-3(14x+9)$
4. $2x(x-7)+3(7-x)$
5. $7y(6-y)+3(y-6)$
6. $11x(x-7)-3(7-x)$

Example 2 Factor by grouping

a. Factor the expression $x^3+7x^2-2x-14$

Solution

$$x^3+7x^2-2x-14 = (x^3+7x^2)+(-2x-14)$$
$$= x^2(x+7)-2(x+7) = (x^2-2)(x+7)$$

PROBLEMS

7. $x^3+6x^2+5x+30$
8. $9x^2+9x-7x-7$
9. $2x^3-6x^2+4x-12$

Example 3 **Factor the polynomial completely**

a. Factor the expression $5x^3 - 55x^2 + 90x$

Solution $5x^3 - 55x^2 + 90x = 5x(x^2 - 11x + 18)$
$= 5x(x-2)(x-9)$

b. Factor the expression $48x^4 - 3x^2$

Solution $48x^4 - 3x^2 = 3x^2(16x^2 - 1)$
$= 3x^2(4x+1)(4x-1)$

PROBLEMS

10. $2x^3 - 16x^2 + 32x$

11. $3x^3 + 18x^2 + 24x$

12. $2x^3 - 4x^2 - 30x$

13. $5x^3 - 20x$

14. $7x^3 - 63x$

15. $3x^3 - 48x$

Example 4 **Solve a polynomial equation**

Solve the equation $4x^3 + 4x^2 - 24x = 0$

Solution
$4x^3 + 4x^2 - 24x = 0$
$4x(x^2 + x - 6) = 0$
$4x(x+3)(x-2) = 0$

$x = 0 \quad x = -3 \quad x = 2$

The solutions of the equation are 0, -3, and 2.

PROBLEMS **Solve the polynomial equation:**

16. $2x^3 + 18x^2 + 40x = 0$

17. $3x^3 + 9x^2 + 6x = 0$

18. $4x^3 + 12x^2 - 72x = 0$

Name: ______________________________ Date: ______________

THE ZEROES OF A QUADRATIC

FLUENCY

1. The roots of $x^2-6x-16=0$ can be found by factoring as

   (1) $\{-16, 6\}$ (3) $\{-2, 8\}$

   (2) $\{-8, 2\}$ (4) $\{6, 16\}$

2. The equation $(2x-3)(x+7)=0$ has a solution set of

   (1) $\{-7, 1\frac{1}{2}\}$ (3) $\{-7, 3\}$

   (2) $\{3, 7\}$ (4) $\{\frac{1}{2}, -3\}$

3. Find the roots of each of the following equations by factoring:

   (a) $x^2-36=0$

   (b) $x^2+12x+27=0$

   (c) $3x^2+5x-2=0$

   (d) $20x^2-10x=0$

   (e) $10x^2+x-21=0$

   (f) $4x^2-16x-84=0$

Name: ______________________________ Date: ________________

QUADRATIC WORD PROBLEMS

Now that we have the Zero Product rule as a method for solving quadratic equations that are **factorable when set equal to zero**, we can also model scenarios that are quadratic in nature and solve them for rational solutions with factoring.

Exercise #1: Consider a rectangle whose area is 45 square feet. If we know that the length is one less than twice the width, then we would like to find the dimensions of the rectangle.

(a) If we represent the width of the rectangle using the variable W, then write an expression for the length of the rectangle, L, in terms of W.

(b) Set up an equation that could be used to solve for the width, W, based on the area.

(c) Solve the equation to find both dimensions. Why is one of the solutions for W not **viable**?

Exercise #2: A square has one side increased in length by two inches and an adjacent side decreased in length by two inches. If the resulting rectangle has an area of 60 square inches, what was the area of the original square? First, draw some possible squares and rectangles to see if you can solve by guess-and-check. Then, solve it algebraically.

We can certainly play around with word problems that involve strictly numbers. For example…

***Exercise* #3:** There are two rational numbers that have the following property: when the product of seven less than three times the number with one more than the number is found it is equal to two less than ten times the number. Find the two rational numbers that fit this description.

And, of course, who can forget our work with **consecutive integers** from the linear unit?

***Exercise* #4:** Find all sets of two consecutive integers such that their product is eight less than ten times the smaller integer.

***Exercise* #5:** Brendon claims that the number five has the property that the product of three less than it with one more than it is the same as the three times one less than it. Show that Brendon's claim is true and algebraically find the other number for which this is true.

Name: ______________________ Date: ____________

QUADRATIC WORD PROBLEMS

FLUENCY

1. The product of two consecutive positive even integers is 14 more than their sum. Set up an equation that can used to find the two numbers and solve it.

2. The length of a rectangle is 4 less than twice the width. The area of the rectangle is 70 square feet. Find the width, w, of the rectangle algebraically. Explain why one of the solutions for w is not viable.

3. Two sets of three consecutive integers have a property that the product of the larger two is one less than seven times the smallest. Set up and solve an equation that can be used to find both sets of integers.

APPLICATIONS

4. A curious patterns occurs in a group of people who all shake hands with one another. It turns out that you can predict the number of handshakes that will occur if you know the number of people.

   If we are in a room of 5 people, we can determine the number of handshakes by this line of reasoning:

   The first person will shake 4 hands (she won't shake her own). The second person will shake 3 hands (he won't shake his own of the hand of the first person, they already shook). The third person will shake 2 hands (same reasoning). The fourth person will shake 1 hand (that of the fifth person). The fifth person will shake 0 hands. So there will be a total of 1+2+3+4=10 handshakes

   (a) Determine the number of handshakes, h, that will occur for each number of people, n, in a particular room.

n (people)	Calculation	h (handshakes)
2		
3		
4		
5	$1+2+3+4=10$	10
6		

   (b) Using knowledge from Algebra II, Prestel proposes the formula $h=\frac{n(n-1)}{2}$ to find the number of handshakes, h, if he knows the number of people. Test the formula and compare with the results you found in (a).

n (people)	$h=\frac{n(n-1)}{2}$	Comparison to (a)
2		
3		
4		
5		
6		

   (c) Assuming Prestel's formula is correct, algebraically determine number of people in a room if there are 66 handshakes that occur.

Name: ______________________________ Date: ________________

FINAL WORK WITH QUADRATIC EQUATIONS

You now have a large variety of ways to solve quadratic equations, i.e. polynomial equations whose highest powered term is x^2. These techniques include **factoring, Completing the Square,** and the **Quadratic Formula**. In each application, it is essential that the equation that we are solving is equal to zero. If it isn't, then some minor manipulation might be needed.

***Exercise* #1:** Solve each of the following quadratic equations using the required method. First, arrange the equations so that they are set equal to zero.

(a) Solve by factoring:

$x^2+5x-12=8x-2$

(b) Solve by Completing the Square.

$x^2-15x+24=-3x+4$

(c) Solve using the Quadratic Formula
Express answers to the nearest tenth.

$x^2-3x+16=5x+15$

(d) Solve using the Quadratic Formula
Express answers in simplest radical form.

$x^2+4x+2=-2x+7$

Name: ______________________________ Date: ________________

FINAL WORK WITH QUADRATIC EQUATIONS

FLUENCY

1. Solve each of the following equations using the method described. Place your final answers in the from asked for.

(a) Solve by factoring:
(Answers are exact)

$2x^2 - 2x + 1 = 4x + 1$

(b) Solve by factoring:
(Answers are exact)

$2x^2 + 5x + 3 = x^2 + 9x + 15$

(c) Solve by Completing the Square
(Round answers to the nearest *tenth*)

$x^2 + 10x + 2 = 2x + 5$

(d)Solve using the Quadratic Formula
(Express answers in simplest radical form)

$2x^2 + 3x - 3 = -3x - 4$

2. Which of the following represents the zeroes of the function $f(x) = x^2 - 4x + 2$?

(1) $\{-1, 2\}$

(3) $\{2-\sqrt{2}, 2+\sqrt{2}\}$

(2) $\{2-2\sqrt{2}, 2+2\sqrt{2}\}$

(4) $\{-1, 4\}$

APPLICATIONS

3. The percent of popcorn kernels that will pop, P, is modeled using the equation:

$P = -0.03T^2 + 25T - 3600$, where T is the temperature in degrees Fahrenheit.

Determine the two temperatures, to the nearest degree Fahrenheit, that result in zero percent of the kernels popping. Use the Quadratic Formula. Show work that justifies your answer. The numbers here will be messy. Use your calculator to help you and carefully write out your work.

REASONING

4. Find the zeroes of the function $y = x^2 - 4x - 16$ by Completing the square. Express your answers in simplest radical form. Graph the parabola using a standard window to see the irrational zeroes.

5. Explain how you can tell that the quadratic function $y = x^2 + 6x + 15$ has no real zeroes *without* graphing the function.

6. Use the Quadratic Formula to determine which of the two functions below would have real zeroes and which would not, then verify by graphing on your calculator using the STANDARD VIEWING WINDOW.

$y = 2x^2 + 3x - 1$ $\qquad\qquad$ $y = x^2 + 2x + 3$

Graph $y = ax^2 + c$

Goal Graph simple quadratic equations

A quadratic equation is a non-linear function that can be written in the standard form $y=ax^2+bx+c$ where $a \neq 0$.

Every quadratic function is U-shaped and is called **parabola.**

The most basic quadratic function in the family of quadratic functions is $y=x^2$, called the **parent function.**

The lowest or highest point of a parabola is called **vertex.**

The vertical line that passes through the vertex and divides the parabola into two symmetric parts is called the **axis of symmetry.**

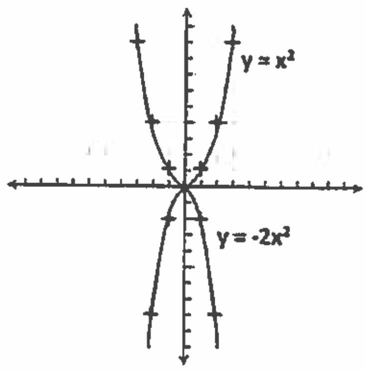
Example 1 Graph $y=ax^2$.

Graph $y = -2x^2$. Compare the graph with the graph $y = x^2$.

Solution 1. Make a table of values for $y=x^2$

x	-2	-1	0	1	2
y	-8	-2	0	-2	-8

2. Plot the point and draw a smooth curve through the points.

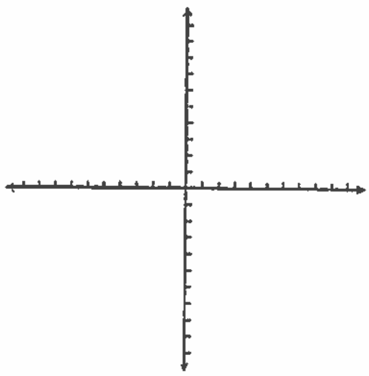


3. Compare: The graph $y=-2x^2$ has the same vertex as $y=x^2$ at (0,0) and the same axis of symmetry. However, the graph of $y=2x^2$ is narrower than $y=x^2$ and it opens down.

PROBLEMS Graph the function. Compare the graph with $y=x^2$

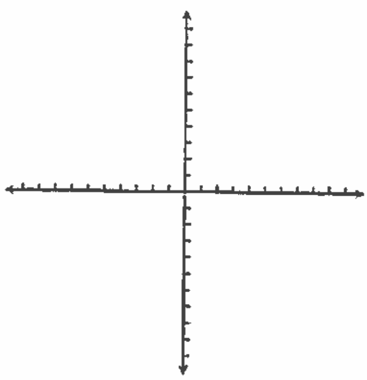
1. $y = -3x^2$

X	-2	-1	0	1	2
Y					



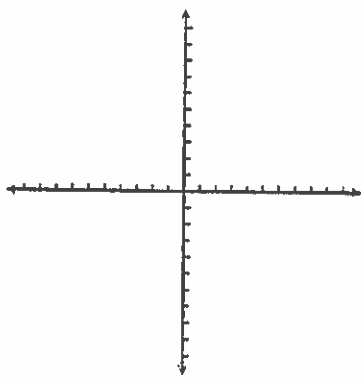
2. $y = -\frac{1}{4}x^2$

X	-4	-2	0	2	4
Y					



3. $y = \frac{1}{3}x^2$

	-4	-3	0	3	4
Y					



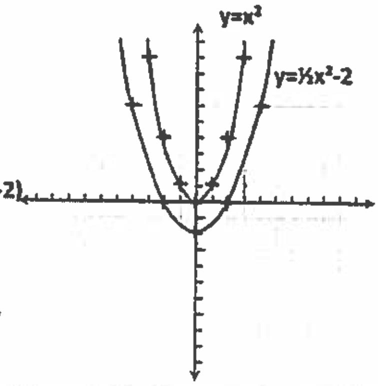
Example 2 Graph y = ax + c².

Graph y = ½ x² -2. Compare the graph with the graph y = x².

Solution 1. Make a table of values for y=½x²-2

x	-4	-2	0	2	2
y	6	0	-2	0	6

2. Plot the point and draw a smooth curve through the points.



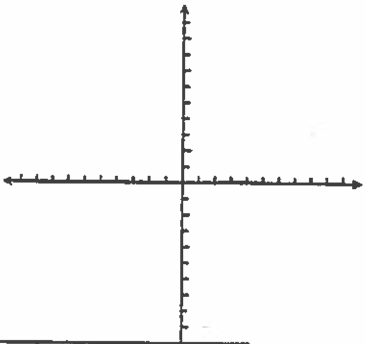
3. Compare: The graph y=½x²-2 has a different vertex as y=x² at (0,-2).

The graph of y=½x²-2 is wider than y=x².

Both graphs open up and have the same axis of symmetry.

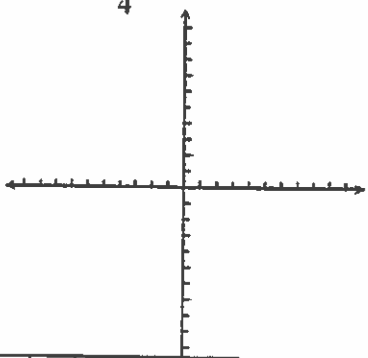
PROBLEMS Graph the function. Compare the graph with y=x²

4. $y = x^2 + 3$



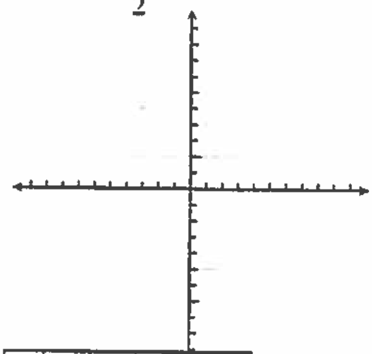
X	-2	-1	0	1	2
Y					

5. $y = \frac{1}{4}x^2 - 2$



X	-4	-2	0	2	4
Y					

6. $y = -\frac{1}{2}x^2 + 1$



X	-4	-2	0	2	4
Y					

Describe how you can use the graph of y=x² to graph the given function.

7. y = x² - 10

8. y = -x² + 3

Analytic Geometry

Name________________________________

Graphing Parabolas from Vertex Form

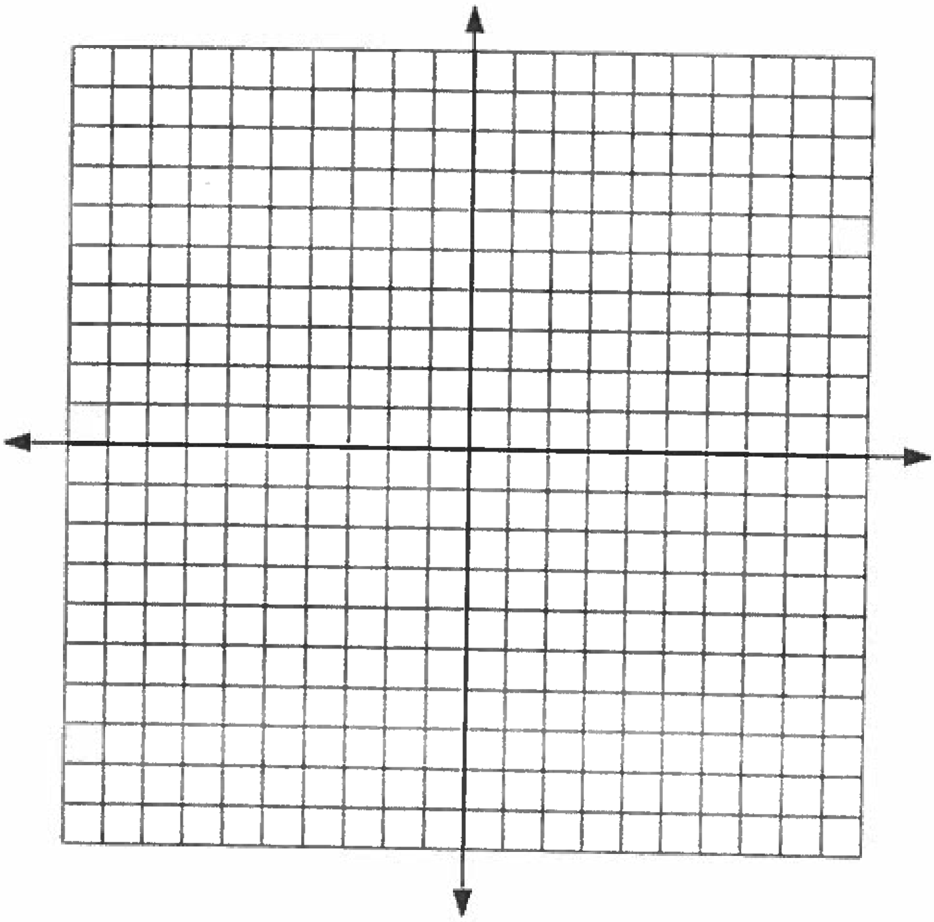
Date________________ Period____

1. Sketch the graphs of these three quadratic relations on the same set of axes.

a) $y = -3x^2$

b) $y = \frac{1}{4}x^2$

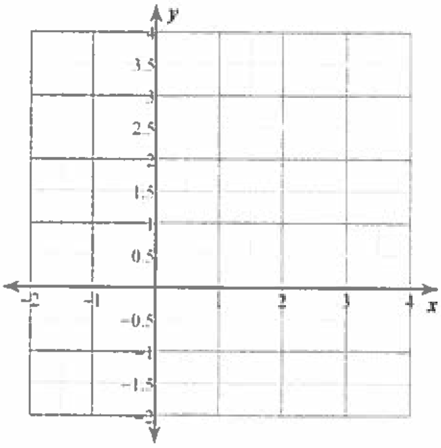
c) $y = -\frac{1}{4}x^2$



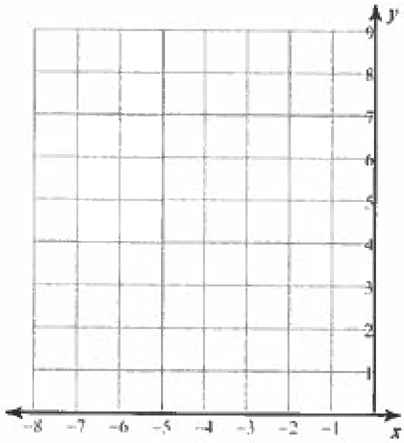
60

Sketch the graph of each function.

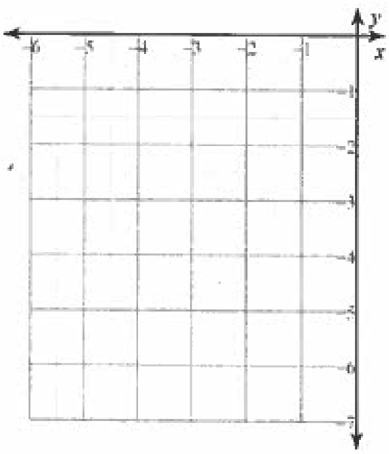
1) $y = (x - 1)^2 - 1$



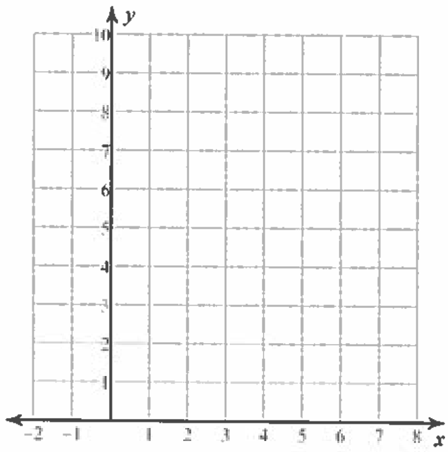
2) $y = (x + 2)^2 + 4$



3) $y = -(x + 3)^2 - 2$

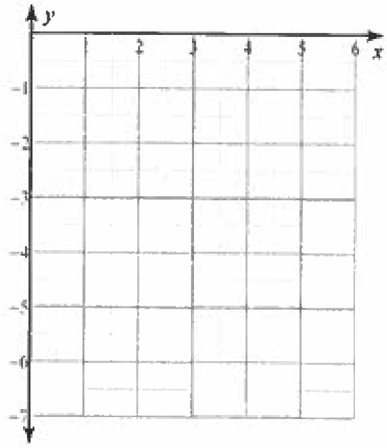


4) $y = 2(x - 2)^2 + 1$

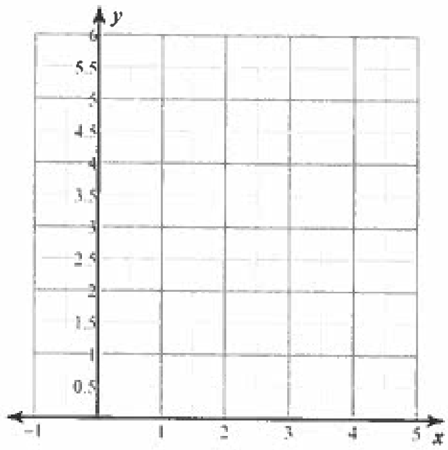


61

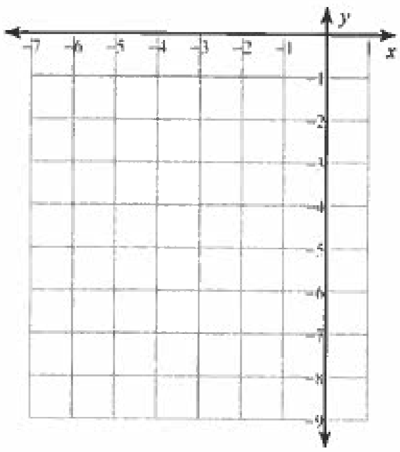
5) $y = -(x-2)^2 - 2$



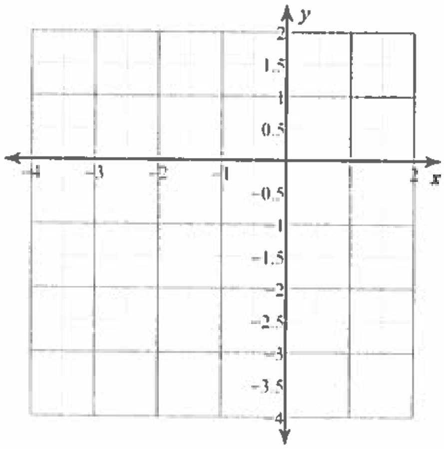
6) $y = (x-3)^2 + 1$



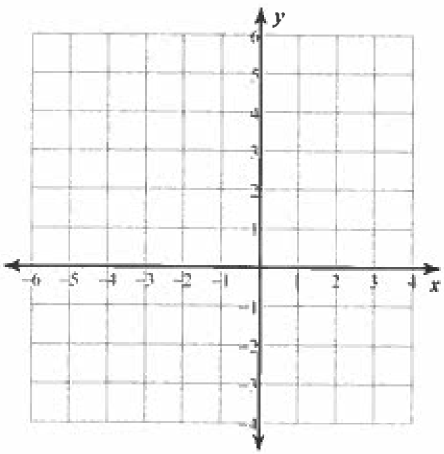
7) $y = -(x+2)^2 - 4$



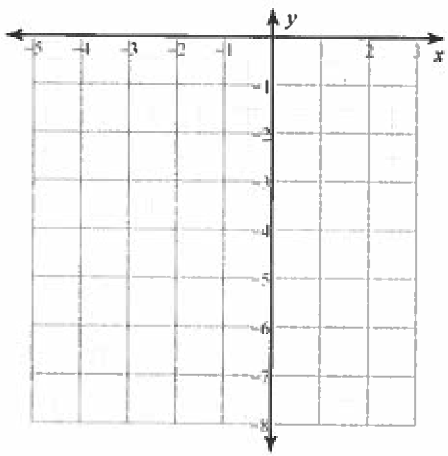
8) $y = (x+1)^2 - 3$



9) $y = 2(x+3)^2 - 3$

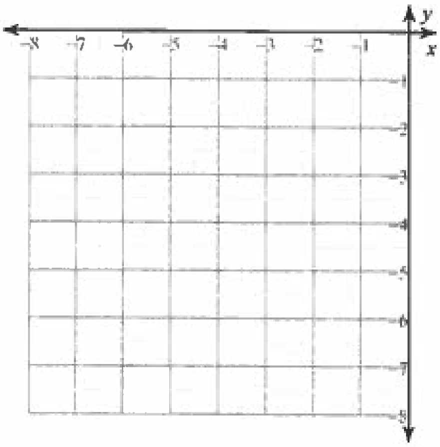


10) $y = -(x+1)^2 - 3$

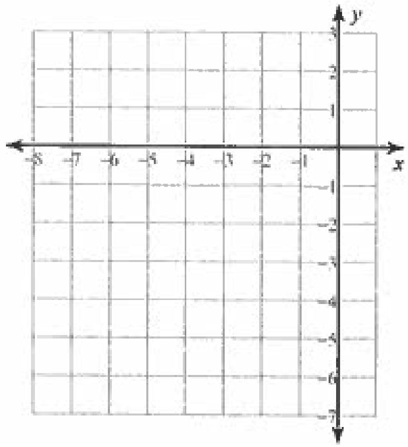


62

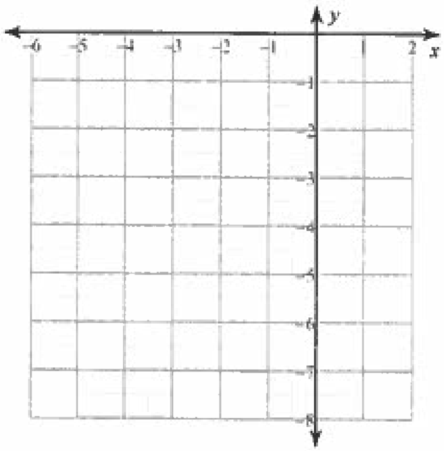
11) $y = -(x+3)^2 - 3$



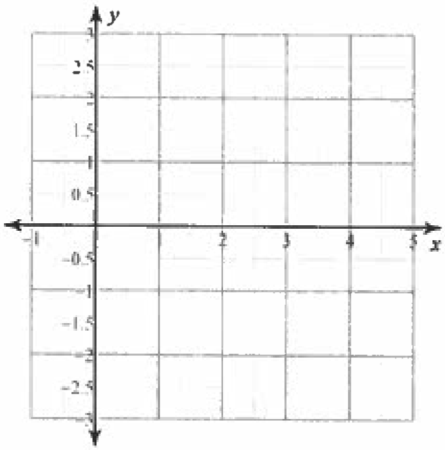
12) $y = -2(x+4)^2 + 2$



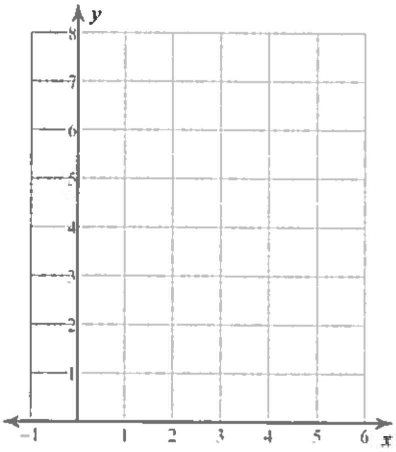
13) $y = -(x+2)^2 - 3$



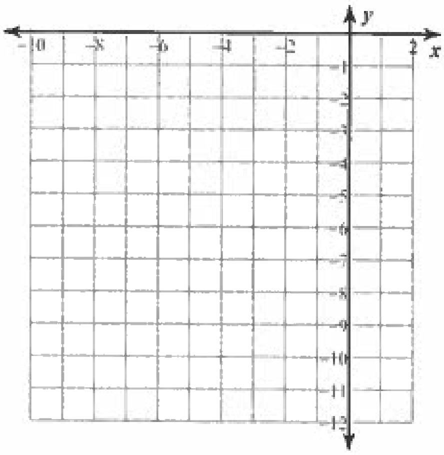
14) $y = -(x-2)^2 + 2$



15) $y = (x-4)^2 + 3$

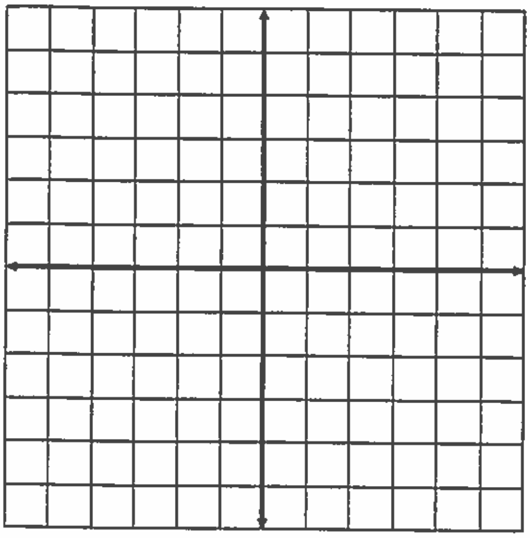


16) $y = -2(x+3)^2 - 3$

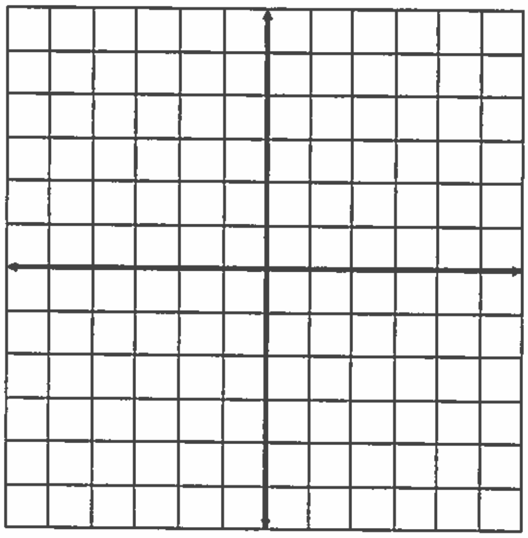


Graphing in Standard Form

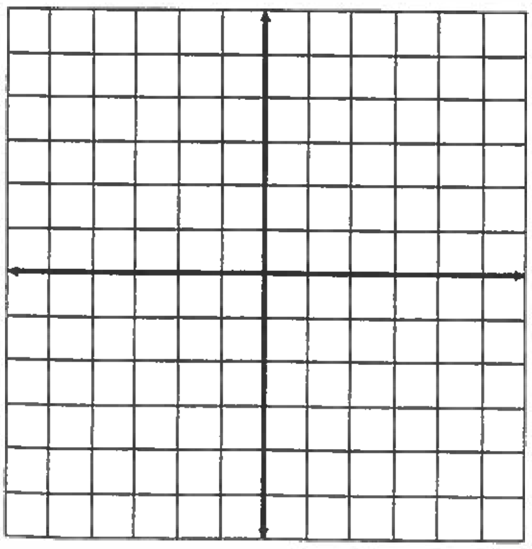
1) $y = x^2 + 6x + 9$



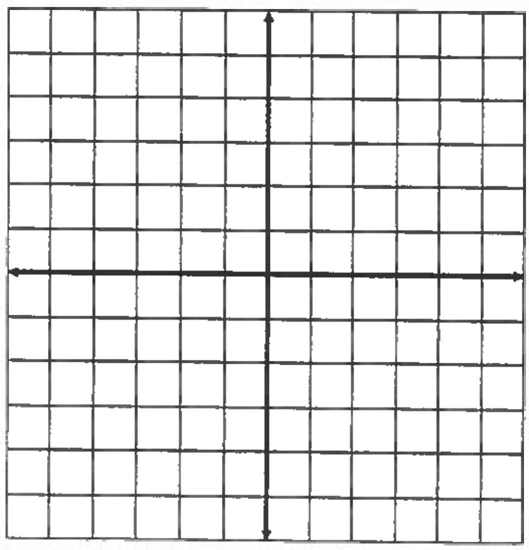
2) $y = x^2 - 2x + 6$



3) $y = x^2 - 8x + 12$



4) $y = 3x^2 - 12x + 9$

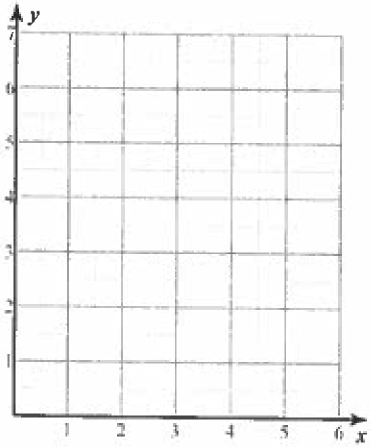


64

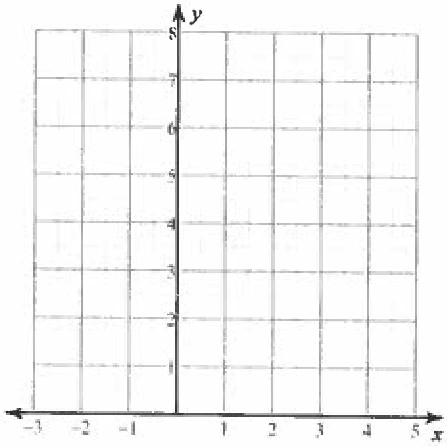
Graphing Parabolas in Standadrd Form

Sketch the graph of each function.

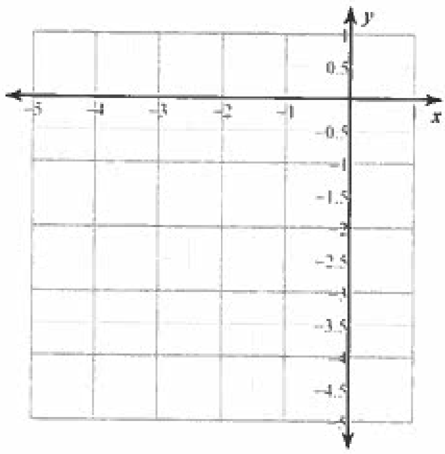
1) $y = x^2 - 4x + 6$



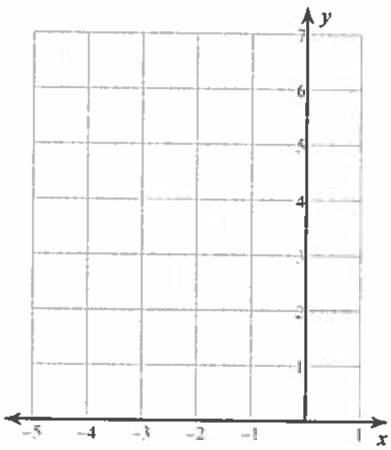
2) $y = x^2 + 2x + 4$



3) $y = x^2 + 4x$

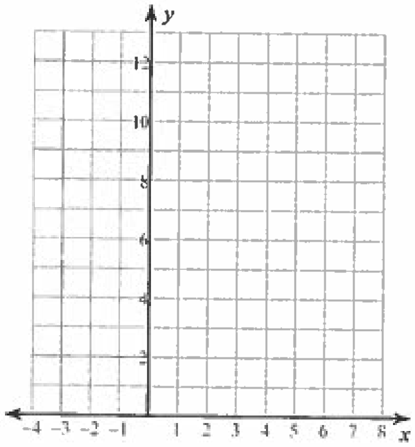


4) $y = x^2 + 2x + 3$

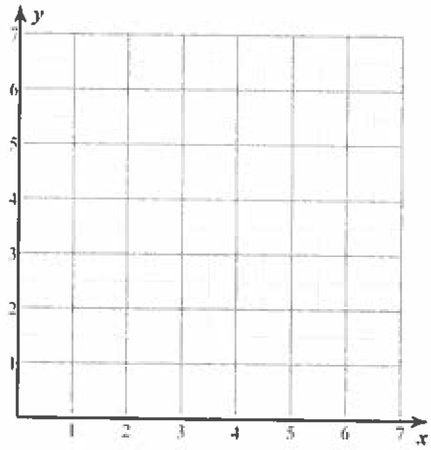


65

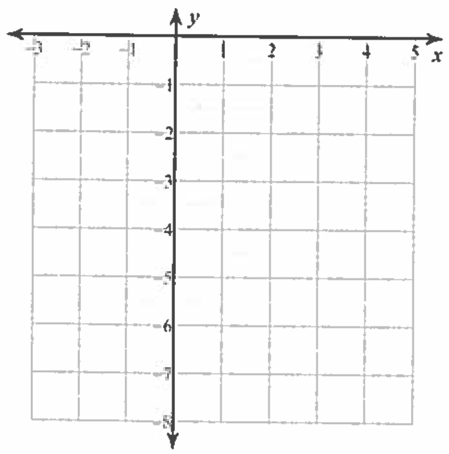
5) $y = 2x^2 - 8x + 12$



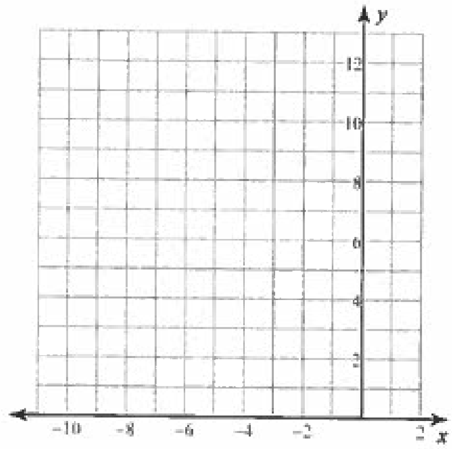
6) $y = x^2 - 8x + 18$



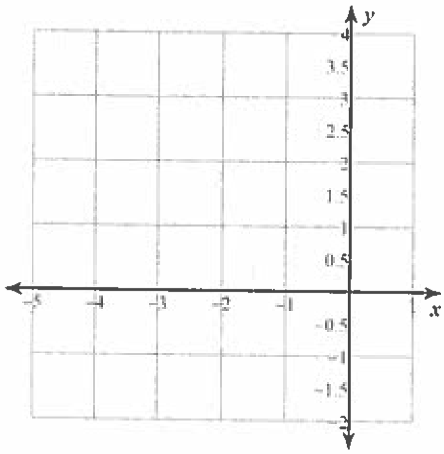
7) $y = -x^2 + 2x - 4$



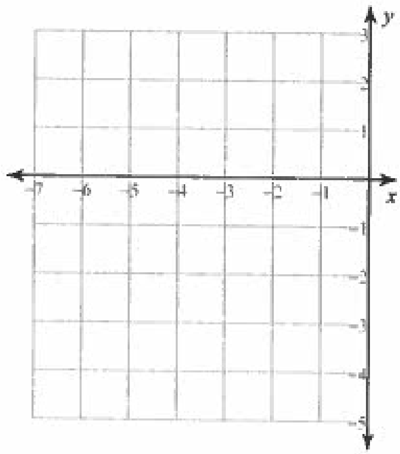
8) $y = 2x^2 + 16x + 36$



9) $y = x^2 + 6x + 8$

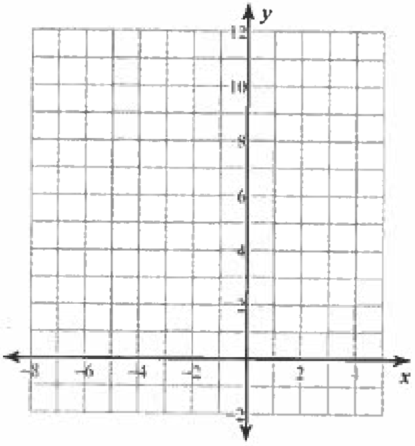


10) $y = x^2 + 8x + 13$

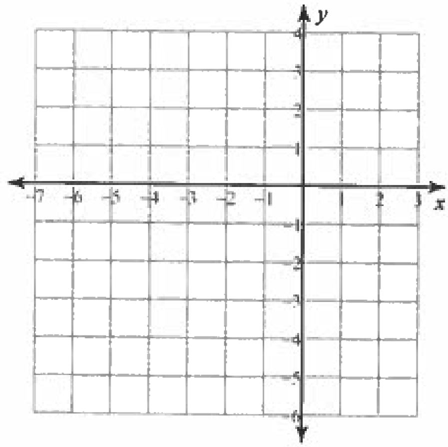


66

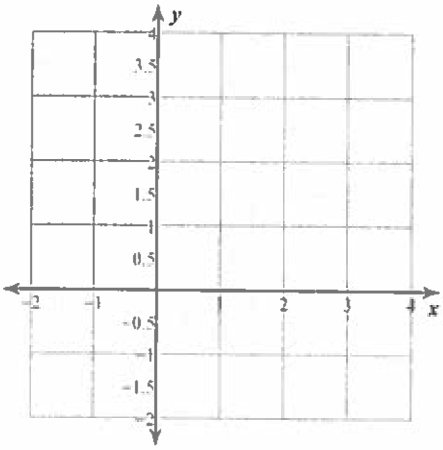
11) $y = 3x^2 + 24x + 47$



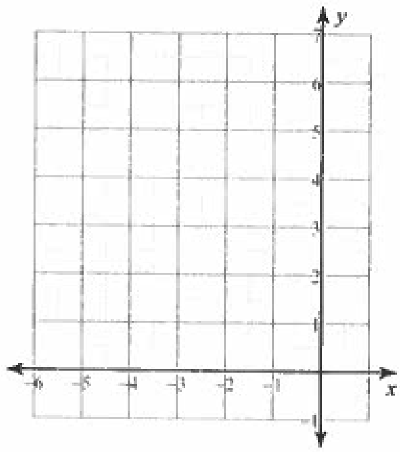
12) $y = -2x^2 - 4x + 1$



13) $y = -\frac{1}{2}x^2 + 2x$



14) $y = x^2 + 8x + 17$

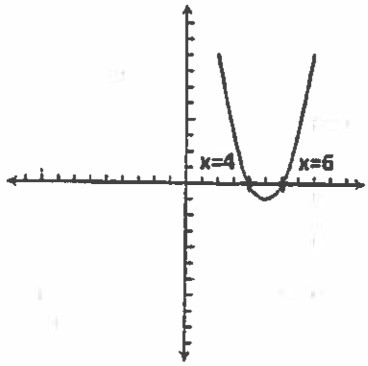


Example 1 Find the zeros of $f(x) = x^2 - 10x + 24$.

Solution Graph the function $x^2 - 10x + 24$

The intercepts are 4 and 6.

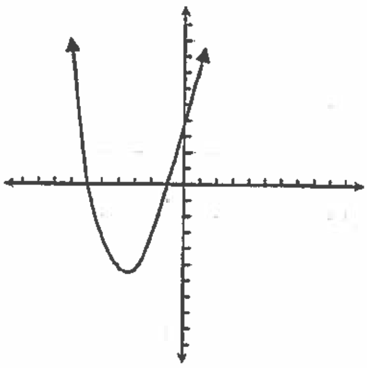
Therefore, the zeroes of the function are 4 and 6.



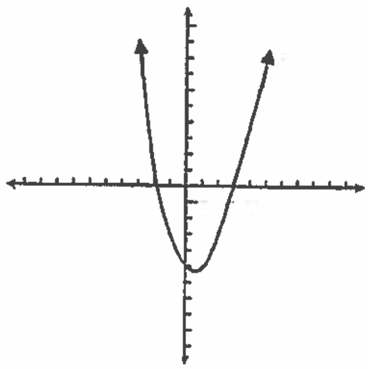
PROBLEMS

Find the zeros of the function.

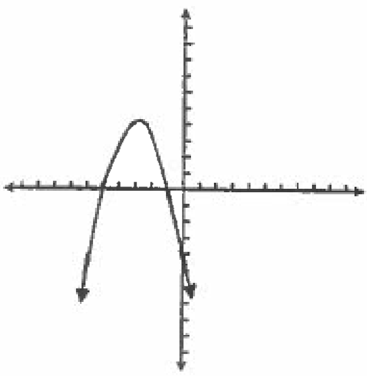
4.



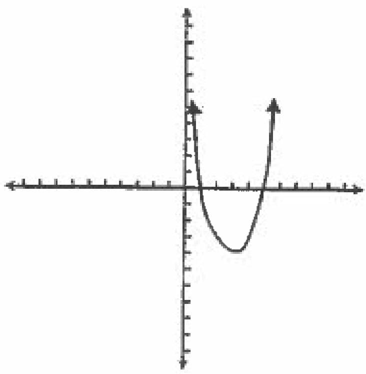
5.



6.



7.



Goal Solve quadratic equations by graphing

A quadratic equation is a non-linear function that can be written in the standard form $y=ax^2+bx+c$ where $a \neq 0$.

The zero of a function is the value for x where the line crosses the x-axis. The corresponding y-value is zero.

Example 1 Graph $x^2-2x=8$.

Solution **STEP 1: Write the equation in standard form.**

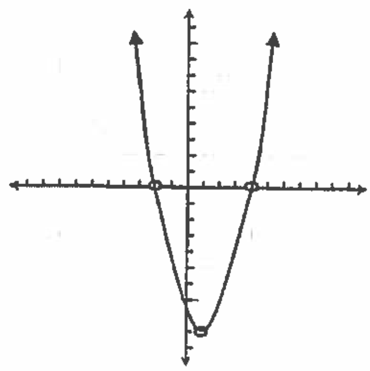
$x^2-2x=8$ **Write original equation**

$x^2-2x-8=0$ **Subtract 8 from both sides**

Step 2: Graph the function $y=x^2-2x-8$

The x-intercepts are -2 and 4.

The corresponding y-values are zero. Therefore, at (-2,0) and (4,0) the graph crosses the x-axis.



Step 3: Check

Plug into original equation

$x^2-2x=8$

$(-2)^2-2(-2)=8$ **Correct**

$4+4 \quad =8$

$(4)^2-2(4)=8$ **Correct**

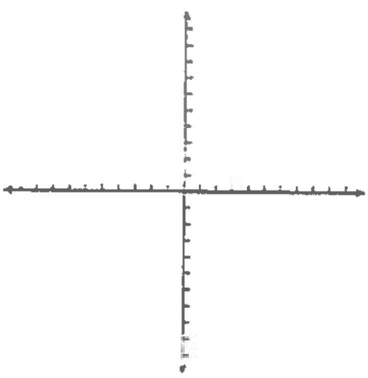
$16 - 8 \quad =16$

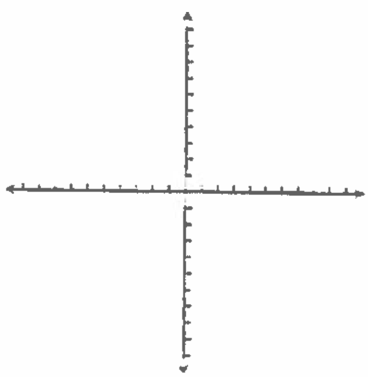
PROBLEMS Solve the equation by graphing (use a graphing calculator).

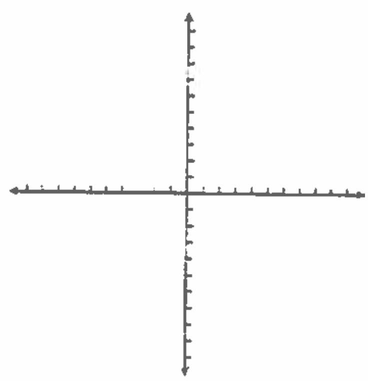
1. $x^2+1x=12$

2. $x^2+6x=-8$

3. $x^2+3x=4$







Graph $y = ax^2 + bx + c$

Goal Graph general quadratic equations

For $y=ax^2+bx+c$ the y-coordinate of the vertex is the minimum value of the function if a is positive and the maximum value of the function if a is negative.

Example 1 Consider the function $y = 3x^2 - 18x + 11$.

a. Find the axis of symmetry of the graph of the function

Solution For the function $y = 3x^2 - 18x + 11$, a = 3 and b = -18.

The x value of the vertex is given by: $x = -\frac{b}{2a} = -\frac{(-18)}{2(3)} = 3$

Therefore the axis of symmetry is x = 3.

b. Find the vertex of the graph of the function

Solution The y-value of the vertex can be found by plugging the x value of the vertex into the original equation:

$y = 3x^2 - 18x + 11 = 3(3)^2 - 18(3) + 11 = -16$

The vertex is (3,-16)

c. Tell whether the function $y = 3x^2 - 18x + 11$ has a minimum value or maximum value.

Solution Because a = 3 (which is a positive number) the graph opens up. Therefore the vertex (3,-16) is a minimum value. The minimum value of the function is -16 (which is the y-value of the function).

PROBLEMS Find the axis of symmetry and the vertex of the graph of the function. Then state whether the vertex is a minimum or a maximum value for the function.

1. $y = 5x^2 + 20x + 9$

2. $y = -2x^2 + 4x - 9$

3. $y = 6x^2 + 12x + 4$

4. $y = -\frac{3}{4}x^2 + 3x + 9$

Example 2 Graph $y = 2x^2 - 8x + 3$.

Solution <u>STEP 1</u>: Determine whether the parabola opens up or down.

Since a is positive (a=2) the parabola opens up.

<u>Step 2</u>: Find and draw the axis of symmetry.

$x = -\frac{b}{2a} = -\frac{(-8)}{2(2)} = 2$ Therefore, draw a vertical line through x=2.

<u>Step 3</u>: Find and plot the vertex.

Plug in 4 for x into the equation and solve for y: $y = 2(2)^2 - 8(2) + 3 = -5$

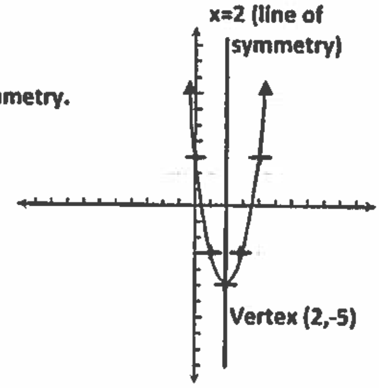
Therefore, the vertex is at (2 , -5). Mark the vertex on the graph.

<u>Step 4</u>: Plot two points. Choose two points less than the x-coordinate of the vertex. Then find the corresponding y-values and plot the points.

x	0	1
y	3	-3

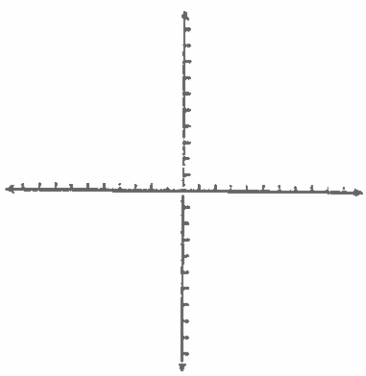
Step 5: Reflect the points plotted in Step 4 about the axis of symmetry.

Step 6: Draw a parabola through the plotted points.

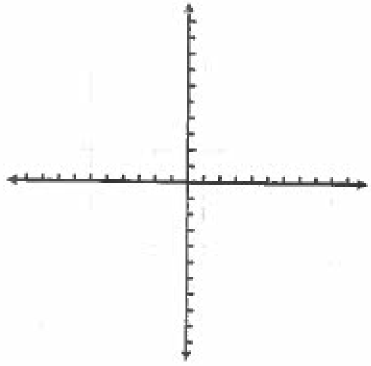


<u>PROBLEMS</u> Graph the function. Label the vertex and axis of symmetry.

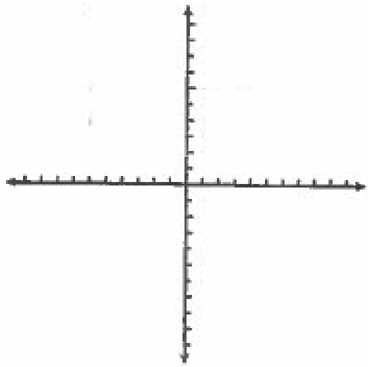
5. $y = x^2 - 4x + 7$



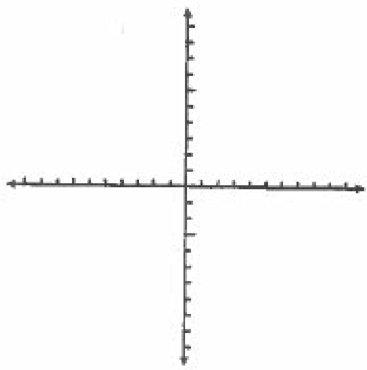
6. $y=-x^2+4x+3$



7. $y=-3x^2-6x+1$



8. $y=x^2-2x+3$



9. $y=\frac{1}{2}x^2+2x+1$